

Goal: motivate stacks as desirable notions of moduli

(from functorial, groupoid, étale perspectives)

Moduli

X : mathematical objects of interest

Def. A moduli space over X is a space M such that

$$\{\text{pts of } M\} \xleftrightarrow{(b_{ij})} \{\text{equiv. classes of } M\}$$

In alg geo, these typically have structures of varieties / schemes.

Ex. (1) $\{\text{lines in } \mathbb{A}^2 \text{ passing through origin}\} \leftrightarrow \mathbb{P}^1$

$$(2) \left\{ C \subset \mathbb{P}^2 \text{ deg } d \text{ curves} \right\} / \sim_{\text{proj. equiv}}$$

$$\begin{aligned} &\leftrightarrow \left\{ p(x, y, z) \text{ deg } d \text{ homo polys} \right\} / \text{PGL}_2(k) \\ &= \mathbb{P}(S_{\text{ym}^d \mathcal{O}_{\mathbb{P}^2}^{\oplus 3}}) / \text{Aut}(\mathbb{P}^2) \end{aligned}$$

$$\left(\begin{array}{l} C \sim C' \text{ proj. equiv} \\ \Leftrightarrow \exists \phi \in \text{Aut}(\mathbb{P}^2) \text{ s.t.} \\ \mathbb{P}^2 \xrightarrow{\phi} \mathbb{P}^2 \\ \uparrow \quad \subset \quad \downarrow \\ C \xrightarrow{\cong} C' \end{array} \right)$$

But all varieties have some cardinality, so b_{ij} 's don't tell us much.

Q. How to improve the notion of moduli?

Ex. Let $\left\{ \begin{array}{l} \text{sm proj conn curves} \\ \text{of genus } g \end{array} \right\} / \sim_{\text{isom}} \leftrightarrow M_g$

and suppose M_g is a scheme.

Wish $\equiv U_g \rightarrow M_g$ s.t. $\forall$ curve C/k that corresponds to $x \in M_g$:

$\left(\begin{array}{l} \text{"universal"} \\ \text{family"} \end{array} \right) \quad C \cong (U_g)_x \times_{k(x)} k$

Improvement #1: moduli as functors

Q. Why functors?

(a) To contain more information, desirable to have data for all base and pullback maps for each morphism b/t bases

(b) $\exists$ Special kind of functors: representable functors

We can hope our moduli functor is representable.

Recall: $F: \text{Sch}^{\text{op}} \rightarrow \text{Set}$ is representable if $\exists X \in \text{Sch}: F \cong h_X = \text{Hom}(-, X)$
(in which case, such X is unique)

Remark. $X \in \text{Sch}$ can be identified as $h_X \in \text{Fun}(\text{Sch}^{\text{op}}, \text{Set})$
by the Yoneda embedding $X \mapsto \text{Fun}(\text{Sch}^{\text{op}}, \text{Set})$.

$\Rightarrow$ The class of moduli functors enlarges the class of schemes!

Representable functors are prevalent in alg geo.

Ex. (1)
$$\left\{ (L, s_0, \dots, s_n) \mid \begin{array}{l} L \text{ LB on } S \text{ globally gen.} \\ \text{by } s_0, \dots, s_n \in T(S, L) \end{array} \right\} / \sim \cong \text{Hom}(S, \mathbb{P}_{\mathbb{Z}}^n)$$

(Hartshorne II.7.1)

(2) (Grassmannian) Fix $0 < k < n$.

$$\{ \mathcal{O}_S^{\oplus n} \rightarrow \mathcal{Q} \text{ rank } k \} / \sim \cong \text{Hom}(S, \text{Gr}(n, k))$$

(3) (Hom scheme) Fix schemes X, Y projective over $\text{Spec } \mathbb{Z}$ with X flat.

$$\text{Hom}_S(X \times S, Y \times S) \cong \text{Hom}(S, \mathcal{H}\text{om}(X, Y))$$

(cf. More general version of Grassmannian, Hilbert, Quot schemes in Chapter 1)

Ex. (moduli functor of smooth curves of genus g)

$$F_{M_g} : \text{Sch}^{\text{op}} \rightarrow \text{Set}$$

$$S \mapsto \left\{ \begin{array}{l} \text{sm proper } \mathcal{C} \rightarrow S \text{ s.t.} \\ \text{all fibers are conn. curves of genus } g \end{array} \right\} / \sim_{S\text{-isom}}$$

(= family of curves)

$$(S' \rightarrow S) \mapsto (F_{M_g}(S) \rightarrow F_{M_g}(S'))$$

$$[\mathcal{C} \rightarrow S] \mapsto [\mathcal{C} \times_S S' \rightarrow S']$$

Q. Is F_{M_g} representable?

IA It is, suppose it is representable by scheme M_g . ($F_{M_g} \cong h_{M_g}$)

$$F_{M_g}(M_g) \cong \text{Hom}(M_g, M_g)$$

$$[U_g \rightarrow M_g] \mapsto \text{id}_{M_g} \quad (U_g : \text{"universal family"})$$

$$\forall \mathcal{C} \rightarrow S \text{ family of curves : } \exists! S \rightarrow M_g \text{ s.t. } \begin{array}{ccc} \mathcal{C} & \xrightarrow{\exists} & U_g \\ \downarrow & \square & \downarrow \\ S & \rightarrow & M_g \end{array}$$

Plugging $S = \text{Spec}(k)$

$$\Rightarrow \{ \text{curves} / k \} \xleftrightarrow{(\text{id})} \{ k\text{-valued points on } M_g \}$$

$$\mathcal{C} \mapsto (\text{Spec}(k) \rightarrow \mathcal{C})$$

$$\& \mathcal{C} \cong (U_g)_x \times_{k(x)} k$$

so our Wish from p.1 is granted!

Fact. Unfortunately, F_{M_g} is not representable.

(so everything in IA is false)

Let's analyze what obstructs F_{M_3} from being representable.

Prop. k alg. closed field, $F: \text{Sch}_k^{\text{op}} \rightarrow \text{Set}$ functor.

Suppose $\exists S$ variety / k , $\epsilon \in F(S)$

s.t. $\left\{ \begin{array}{l} (a) \forall s \in S(k) = \text{Hom}_k(\text{Spec}(k), S) : E_s = e \in F(k) \text{ independent of } s \\ (b) E \text{ not a pullback of any elt in } F(k) \\ \quad (\text{via } S \rightarrow \text{Spec}(k)) \end{array} \right.$

Then F is not representable.

pf. Assume $F = \text{Hom}(-, X)$.

$$V_S \in S(k) : \begin{array}{ccccc} S_{\text{Spec}(k)} & \xrightarrow{s} & S & \xrightarrow{\varepsilon} & X \\ & \searrow \scriptstyle \cup & & \nearrow \scriptstyle \varepsilon & \\ & & & & \end{array}$$

Let $e(\cdot) = x \in X$. (closed pt)

$$\mathcal{E}(S) \subseteq \overline{\mathcal{E}(S)} = \overline{\mathcal{E}\left(\underbrace{\left\{\text{closed of } S^{\text{res}}\right\}}_{\substack{\uparrow \\ \text{b/c dense subset of } S}}\right)} = \overline{\{x\}} = \{x\}.$$

Also reduced subscheme on $\{x\} \subseteq X$ is $\text{Spec}(k)$, so

$$S \rightarrow X \text{ splits } S \rightarrow \text{Spec}(k) \xrightarrow{\alpha} X.$$

$\exists \in F(S)$ pullback of $\alpha \in F(k)$, contradiction.



Claim. F_{M_3} is not represented by a scheme $/k=\bar{k}$.

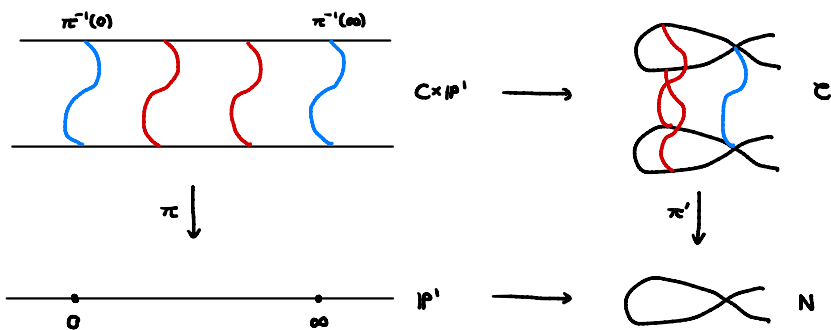
Prop. If $F_{M_3} \cong h_X$ for some $X \in \text{Sch}_k$,

$\text{Sch}_k^{\text{op}} \rightarrow \text{Sch}^{\text{op}} \xrightarrow{F_{M_3}} \text{Set}$ also represented by $X \in \text{Sch}_k$.
(forgetful functor)

So suffices to show $\text{Sch}_k^{\text{op}} \rightarrow \text{Set}$ not representable.

Now take $\begin{cases} \text{curve } C/k \text{ of genus } g \text{ with non-trivial } \alpha \in \text{Aut}(C) \\ N = \mathbb{P}^1_k \text{ with } 0, \infty \text{ glued together (nodal cubic curve)} \end{cases}$

Take trivial family $\pi: C \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ and glue $\pi^{-1}(0)$, $\pi^{-1}(\infty)$ via α to get a new family of curves $\pi': C \rightarrow N$.



The variety N/k & $[C \rightarrow N] \in F_{M_3}(N)$ satisfies the conditions in Prop: $\begin{cases} \text{all fibers of } C \rightarrow N \text{ are } C \Rightarrow (a) \\ C \rightarrow N \text{ is not a trivial family} \Rightarrow (b) \end{cases}$

So by Prop, F_{M_3} is not representable. $\square$

Upshot Non-trivial automorphisms provide an obstruction on representability of moduli functors.

Thus, to improve our notion of moduli, we need to include the data of automorphisms.
(i.e. make it close to being representable)

Improvement #2: moduli as stacks

Candidate for adding automorphism data: $F: \text{Sch}^{\text{op}} \rightarrow \text{Groupoids}$
($T \mapsto \text{Fam}_T$)

$\text{Fam}_T = \text{groupoid}$ s.t. objects are families over T (without taking iso classes)
and morphisms are iso b/t families

This construction is not so clean b/c:

(1) Groupoids is a "2-category" ($\exists$ morphisms b/t morphisms)

(2) $\forall S'' \xrightarrow{f} S' \xrightarrow{g} S : F(f) \circ F(g) \neq F(g \circ f)$

($\exists$ natural isomorphism b/t these functors

$\text{Fam}_S \rightarrow \text{Fam}_{S''}$, but manually have to fix them)

$\Rightarrow F$ is not even a functor (but a pseudo-functor)

So instead, we encode the automorphism data in a massive category $\mathcal{X}$ over Sch .

Construction:

- category $\mathcal{X}$

$\left\{ \begin{array}{l} \text{objects: els of } \bigsqcup_S \text{Fam}_S \end{array} \right.$

$\left\{ \begin{array}{l} \text{morphisms: If } a \in \text{Fam}_S \text{ and } b \in \text{Fam}_T, \text{ morphism } a \rightarrow b \text{ is} \\ f: S \rightarrow T \text{ together with isomorphism } a \xrightarrow{\cong} f^*b. \end{array} \right.$

- functor $\mathcal{X} \rightarrow \text{Sch}$ ($a \in \text{Fam}_S \mapsto S$)

Ex. (moduli stack of smooth curves of genus g)

category $\mathcal{M}_g \left\{ \begin{array}{l} \text{objects: family of curves } \mathcal{C} \rightarrow S \\ \text{morphisms: } (\mathcal{C}' \rightarrow S') \rightarrow (\mathcal{C} \rightarrow S) = \begin{array}{ccc} \mathcal{C}' & \xrightarrow{\alpha} & \mathcal{C} \\ \downarrow & \square & \downarrow \\ S' & \xrightarrow{f} & S \end{array} \end{array} \right.$

with functor $\mathcal{M}_g \rightarrow \text{Sch}$ ($(\mathcal{C} \rightarrow S) \mapsto S$)

Def. (not precise)

prestack $\approx$ functor $p: \mathcal{X} \rightarrow \text{Sch}$ s.t. $\exists!$ pullbacks

stack = prestack s.t. $\exists!$ gluing (cf. sheaf)

Remark. (1) The fiber category $\mathcal{X}(S)$ consists of objects over S and morphisms over id_S . If $\mathcal{X}$ is built from groupoids F_{am_S} as above construction, $\mathcal{X}(S) = F_{am_S}$.

(2) (moduli functor as moduli stack)

Any moduli functor $F: \text{Sch}^{\text{op}} \rightarrow \text{Set}$ can be identified as a moduli stack as follows:

$$\text{Ob}(\mathcal{X}_F) = \{(S, a) \mid S \in \text{Sch}, a \in F(S)\}$$

$$\text{Mor}_{\mathcal{X}_F}((S, a), (S', b)) = \{S' \xrightarrow{f} S \mid a = f^*b\}$$

Also the groupoid $\mathcal{X}_F(S)$ only has identity morphisms.

$\Rightarrow$ The class of moduli stacks enlarges the class of moduli functors, and moreover moduli functors are stacks with no non-trivial automorphisms.

Why do we use étale topology?

étale morphisms $\approx$ local diffeomorphisms (in diff geo)

(1) more étale nbhds than Zariski open nbhds

Ex. $C: y^2 = x^2(x-1)$

$$C': \text{Spec } k[x, y, t]_t / (y^2 - x^2 + x^2 t, t^2 - x + 1) \quad (\text{taking square root of } x-1)$$

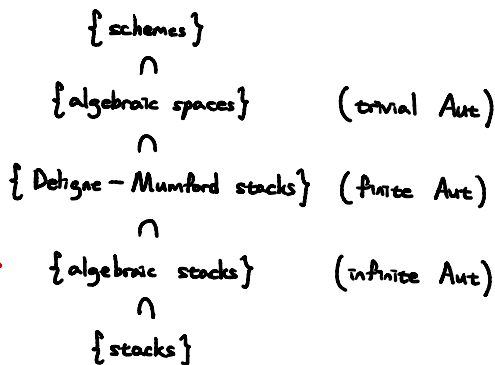
$$\Rightarrow C' \rightarrow C \text{ reducible étale nbhd } (y^2 - x^2 + x^2 = (y - xt)(y + xt))$$

(2) Étale cohomology

(3) Artin approximation

(4) Descent

Hierarchy of objects of interest



defined with representability
conditions

Schedule

Time	Material
Feb-Mar	Ch 2-3
Apr	Ch 4
May-mid June	Ch 5
late June-Aug	Ch 6-7
end of Aug	special topics